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A New Bi-cubic Triangular Gregory Patch

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***Abstract*—A basic problem may be often encountered when people construct smooth surface that interpolates the given curve meshes. In general, object surface can be divided into triangular and quadrangular areas by the given curve meshes. The problem is that while the methods satisfy the continuity and interpolation requirements, they often fail to produce pleasing shapes, or in reverse to the condition. In this paper, we propose a new triangular Gregory patch which is a variant of quadrangular Gregory patch and Bézier triangle. It owns more simple mathematical formula and can make G^1 continuity more easily controlled.**

I. Introduction

It is a problem to construct a surface through the given data points in many application areas such as geological modeling, medical imaging, scientific visualization, geometric modeling and so on. In general, local data points are used to construct portion of the surface by local schemes and the global scheme is a combination of all the local schemes.

In CAGD (Computer Aided Geometric Design) and CG (Computer Graphics), Tensor-product B-spline surface [1] [2], Coons-Gordon patch [1] and Gregory patch [3] [4] as local schemes work well for modeling surfaces based on quadrangular control nets, but they are not sufficient for more general topologies. Because traditional triangular Gregory patch [5] [6] and Bézier triangle [1] [2] are known to have free and flexible performance of shape, they can represent arbitrary topologies. However, when the given data points cannot be always interpolated with a parametric continuous surface, the continuity conditions have to relaxed to G continuity [2]—at least G^1 continuity. Even so, traditional triangular Gregory patch and Bézier triangle cannot guarantee to satisfy the G^1 continuity conditions in any case.

In this paper, we propose a new bi-cubic triangular Gregory patch scheme which is a local, parametric, triangular and interpolatory scheme. The new scheme is sufficient for arbitrary topologies and satisfy the G^1 continuity conditions in any triangular and quadrangular curve meshes case.

In section II, we define the new triangular Gregory patch by modifying traditional triangular Gregory patch scheme. In section III, at first, G^1 continuity conditions among the new triangular Gregory patches are derived. Then, G^1 continuity conditions between quadrangular Gregory patch and the new triangular Gregory patch are derived. In section IV, the results of deduction are discussed.

II. The New Triangular Gregory Patch

A. Traditional Triangular Gregory Patch

Traditional triangular Gregory patch [5] [6] is geometrically defined by a set of points $\mathbf{G} = \{ \mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_{01}, \mathbf{P}_{10}, \mathbf{P}_{12}, \mathbf{P}_{21}, \mathbf{P}_{20}, \mathbf{P}_{02}, \mathbf{q}_{01}, \mathbf{q}_{10}, \mathbf{q}_{12}, \mathbf{q}_{21}, \mathbf{q}_{20}, \mathbf{q}_{02} \}$. In general, $\mathbf{G}$ is called Gregory control nodes or Gregory control net shown in Figure 1. Each point of $\mathbf{G}$ is called control point. For example, $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2\}$ are corner control points, $\{\mathbf{P}_{01}, \mathbf{P}_{10}, \mathbf{P}_{12}, \mathbf{P}_{21}, \mathbf{P}_{20}, \mathbf{P}_{02}\}$ are boundary control points and $\{\mathbf{q}_{01}, \mathbf{q}_{10}, \mathbf{q}_{12}, \mathbf{q}_{21}, \mathbf{q}_{20}, \mathbf{q}_{02}\}$ are interior control points. All the control points can be divided into three control units.

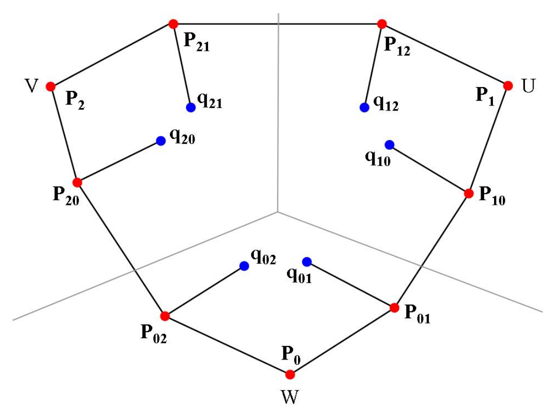

Fig. 1. Triangular Gregory control net

Let $\mathbf{G}_T(u,v)$ be the function which gives the location of a point from its barycentric coordinates (u,v) in the triangular patch. Then $\mathbf{G}_T(u,v)$ can be defined as Equation (1):

$$\begin{aligned}\mathbf{G_T}(u,v) \;=\;& w^3\mathbf{P}_0 + u^3\mathbf{P}_1 + v^3\mathbf{P}_2 \\ &+ 3uv(1-w)(u\mathbf{P}_{12} + v\mathbf{P}_{21}) \\ &+ 3vw(1-u)(v\mathbf{P}_{20} + w\mathbf{P}_{02}) \\ &+ 3wu(1-v)(w\mathbf{P}_{01} + u\mathbf{P}_{10}) \\ &+ 12uvw(w\mathbf{Q}_0 + u\mathbf{Q}_1 + v\mathbf{Q}_2)\end{aligned} \quad (1)$$

Here, $u+v+w=1$ and $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$ are the blending functions of two interior control points that are corresponding to one

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corner control point in the same control unit, respectively:

$$\mathbf{Q}_0 = \frac{u\mathbf{q}_{01} + v\mathbf{q}_{02}}{u+v}$$
$$\mathbf{Q}_1 = \frac{v\mathbf{q}_{12} + w\mathbf{q}_{10}}{v+w} \qquad (2)$$
$$\mathbf{Q}_2 = \frac{w\mathbf{q}_{20} + u\mathbf{q}_{21}}{w+u}$$

In Equation (1), w^3, u^3, v^3 are the respective coefficient of corner control points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ and they are cubic. While, the other control points have quartic coefficients. Because of $w+u+v=1$, w^3, u^3, v^3 can be substituted by $w^3(w+u+v)$, $u^3(w+u+v)$, $v^3(w+u+v)$. Then the bi-quartic Equation (3) can be obtained.

$$\begin{aligned} \mathbf{G_T}(u,v) \;=\;& w^4\mathbf{P}_0 + u^4\mathbf{P}_1 + v^4\mathbf{P}_2 \\ &+ 4u^3v(\frac{3\mathbf{P}_{12}+\mathbf{P}_1}{4}) \\ &+ 6u^2v^2(\frac{3\mathbf{P}_{21}+3\mathbf{P}_{12}}{6}) \\ &+ 4uv^3(\frac{3\mathbf{P}_{21}+\mathbf{P}_2}{4}) \\ &+ 4v^3w(\frac{3\mathbf{P}_{20}+\mathbf{P}_2}{4}) \\ &+ 6v^2w^2(\frac{3\mathbf{P}_{02}+3\mathbf{P}_{20}}{6}) \\ &+ 4vw^3(\frac{3\mathbf{P}_{02}+\mathbf{P}_0}{4}) \\ &+ 4w^3u(\frac{3\mathbf{P}_{01}+\mathbf{P}_0}{4}) \\ &+ 6w^2u^2(\frac{3\mathbf{P}_{10}+3\mathbf{P}_{01}}{6}) \\ &+ 4wu^3(\frac{3\mathbf{P}_{10}+\mathbf{P}_1}{4}) \\ &+ 12uvw(w\mathbf{Q}_0 + u\mathbf{Q}_1 + v\mathbf{Q}_2) \end{aligned} \qquad (3)$$

When $\mathbf{q}_{01} = \mathbf{q}_{02}$, $\mathbf{q}_{12} = \mathbf{q}_{10}$ and $\mathbf{q}_{20} = \mathbf{q}_{21}$, Equation (3) is a bi-quartic Bézier triangle. Therefore, the traditional bi-cubic triangular Gregory patch is the expansion form of bi-quartic Bézier triangle.

B. The New Triangular Gregory Patch

A quadrangular Gregory patch has a control net shown in Figure 2. Similar to triangular Gregory patch, one corner control point, two adjacent boundary control points and two corresponding interior control points construct one control unit. Therefore, the control net can be divided into four control units. In fact, no matter Gregory patch is quadrangular or triangular, each corner control point is a common point of more than two meshes, each boundary control point is a common point of two adjacent meshes, and all interior control points lie in the mesh. In Figure (2), each boundary control point is corresponding to only one interior control point.

If one control unit is removed from quadrangular Gregory control net, three variables expression is used and the marks of control points are changed, we can obtain a triangular Gregory control net shown in Figure 1. Similar to quadrangular Gregory patch, when $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$ are blending functions of two interior control points that are corresponding to one corner control point, Equation (2) can be obtained. Assuming $\mathbf{Q}$ is barycentric combination of $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$, because of $u+v+w=1$, $\mathbf{Q} = \frac{w\mathbf{Q}_0 + u\mathbf{Q}_1 + v\mathbf{Q}_2}{w+u+v} = w\mathbf{Q}_0 + u\mathbf{Q}_1 + v\mathbf{Q}_2$. Therefore, a single new bi-cubic triangular Gregory patch $\mathbf{G_{NT}}(\mathbf{u}, \mathbf{v})$ can be defined as a bi-cubic Bézier triangle like Equation (4):

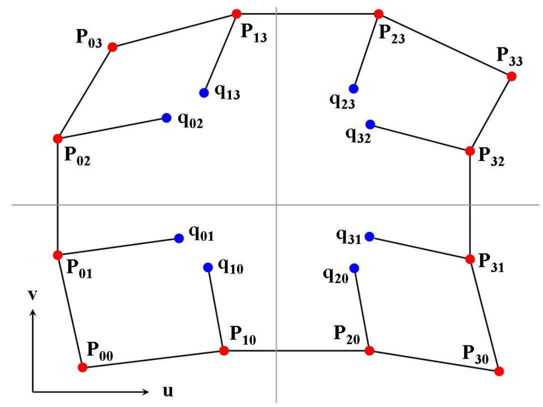

Fig. 2. Quadrangular Gregory control net

$$\begin{aligned} \mathbf{G_{NT}}(u,v) \;=\;& w^3\mathbf{P}_0 + u^3\mathbf{P}_1 + v^3\mathbf{P}_2 \\ &+ 3w^2u\mathbf{P}_{01} + 3w^2v\mathbf{P}_{02} \\ &+ 3u^2w\mathbf{P}_{10} + 3u^2v\mathbf{P}_{12} \\ &+ 3v^2w\mathbf{P}_{20} + 3v^2u\mathbf{P}_{21} \\ &+ 6uvw\mathbf{Q} \end{aligned} \qquad (4)$$

If raising the degree of Equation (4) from cubic to quartic, we can get:

$$\begin{aligned} \mathbf{G_{NT}}(u,v) \;=\;& w^4\mathbf{P}_0 + u^4\mathbf{P}_1 + v^4\mathbf{P}_2 \\ &+ 4u^3v(\frac{3\mathbf{P}_{12}+\mathbf{P}_1}{4}) \\ &+ 6u^2v^2(\frac{3\mathbf{P}_{21}+3\mathbf{P}_{12}}{6}) \\ &+ 4uv^3(\frac{3\mathbf{P}_{21}+\mathbf{P}_2}{4}) \\ &+ 4v^3w(\frac{3\mathbf{P}_{20}+\mathbf{P}_2}{4}) \\ &+ 6v^2w^2(\frac{3\mathbf{P}_{02}+3\mathbf{P}_{20}}{6}) \\ &+ 4vw^3(\frac{3\mathbf{P}_{02}+\mathbf{P}_0}{4}) \\ &+ 4w^3u(\frac{3\mathbf{P}_{01}+\mathbf{P}_0}{4}) \\ &+ 6w^2u^2(\frac{3\mathbf{P}_{10}+3\mathbf{P}_{01}}{6}) \\ &+ 4wu^3(\frac{3\mathbf{P}_{10}+\mathbf{P}_1}{4}) \\ &+ 12uvw(w\mathbf{Q'}_0 + u\mathbf{Q'}_1 + v\mathbf{Q'}_2) \end{aligned} \qquad (5)$$

Here, $\mathbf{Q'}_0, \mathbf{Q'}_1, \mathbf{Q'}_2$ are blending functions of two boundary control points and two corresponding interior control points in one control unit, respectively:

$$\begin{aligned}\mathbf{Q'}_0 &= \frac{3\mathbf{P}_{01}+3\mathbf{P}_{02}+6\mathbf{Q}_0}{12}\\ \mathbf{Q'}_1 &= \frac{3\mathbf{P}_{12}+3\mathbf{P}_{10}+6\mathbf{Q}_1}{12}\\ \mathbf{Q'}_2 &= \frac{3\mathbf{P}_{20}+3\mathbf{P}_{21}+6\mathbf{Q}_2}{12}\end{aligned} \tag{6}$$

By comparing Equation (3) with Equation (5), we are able to obtain the following characters. For the same control net, the patch defined by Equation (3) and the patch defined by Equation (5) have the same boundary curves. Especially:

- When

$$\begin{aligned}\mathbf{q}_{01}+\mathbf{q}_{02} &= \mathbf{P}_{01}+\mathbf{P}_{02}\\ \mathbf{q}_{10}+\mathbf{q}_{12} &= \mathbf{P}_{10}+\mathbf{P}_{12}\\ \mathbf{q}_{20}+\mathbf{q}_{21} &= \mathbf{P}_{20}+\mathbf{P}_{21}\end{aligned} \tag{7}$$

  Equation (3) and Equation (5) have at least one common intersecting point and the coordinates of this point are: $\mathbf{G}_T(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = \mathbf{G}_{NT}(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.
- When

$$\begin{aligned}\mathbf{q}_{01} &= \mathbf{q}_{02}\\ \mathbf{q}_{10} &= \mathbf{q}_{12}\\ \mathbf{q}_{20} &= \mathbf{q}_{21}\end{aligned} \tag{8}$$

  Equation (3) and Equation (5) define two bi-quartic Bézier triangles. Three blending interior control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$ in Equation (3) have no relation with points $\frac{\mathbf{P}_{01}+\mathbf{P}_{02}}{2}$, $\frac{\mathbf{P}_{10}+\mathbf{P}_{12}}{2}$ and $\frac{\mathbf{P}_{20}+\mathbf{P}_{21}}{2}$. While three blending interior control points $\mathbf{Q'}_0$, $\mathbf{Q'}_1$, $\mathbf{Q'}_2$ in Equation (5) converge on points $\frac{\mathbf{P}_{01}+\mathbf{P}_{02}}{2}$, $\frac{\mathbf{P}_{10}+\mathbf{P}_{12}}{2}$ and $\frac{\mathbf{P}_{20}+\mathbf{P}_{21}}{2}$, respectively. Therefore, the new triangular Gregory patch can make the N-side problem [7] [8] and lathy triangular region interpolation problem be more easily solved.
- When

$$\begin{aligned}\mathbf{q}_{01} = \mathbf{q}_{02} &= \frac{\mathbf{P}_{01}+\mathbf{P}_{02}}{2}\\ \mathbf{q}_{10} = \mathbf{q}_{12} &= \frac{\mathbf{P}_{10}+\mathbf{P}_{12}}{2}\\ \mathbf{q}_{20} = \mathbf{q}_{21} &= \frac{\mathbf{P}_{20}+\mathbf{P}_{21}}{2}\end{aligned} \tag{9}$$

  Equation (3) and Equation (5) define the same patch.

III. G^1 Continuity Conditions

A G^1 continuous surface can be characterized as follows: at any point of the surface, a unique normal is defined [2]. In order to insure global G^1 continuity, we need to enforce continuity to skirt along the common boundary curve of any pair of adjacent patches. In this section, we derive G^1 continuity conditions in two kinds of situation, respectively.

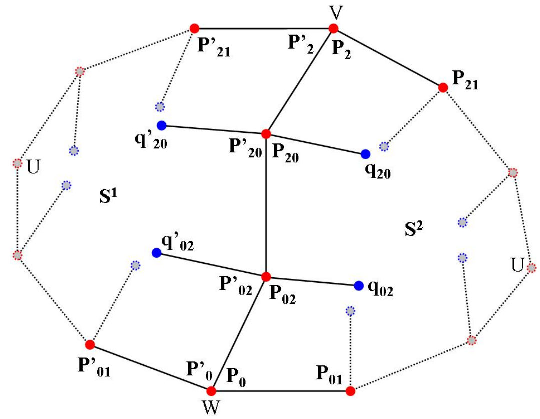

Fig. 3. Two adjacent triangular Gregory patches

A. Among Triangular Gregory Patches

When two Gregory patches are adjacent shown in Figure 3, they have the common boundary curve which is defined by $\{\mathbf{P}_0 = \mathbf{P'}_0, \mathbf{P}_{02} = \mathbf{P'}_{02}, \mathbf{P}_{20} = \mathbf{P'}_{20}, \mathbf{P}_2 = \mathbf{P'}_2\}$.

In order to describe more easily, we transform Gregory patch into two variables (u,v) expression. If $\mathbf{S_u}$ is the partial derivative of u direction and $\mathbf{S_v}$ is the partial derivative of v direction, the necessary and sufficient condition that two adjacent Gregory patches $\mathbf{S^1}$ and $\mathbf{S^2}$ satisfied the G^1 continuity can be expressed as Equation (10):

$$\mathbf{S^2_u}(0,v) = k(v)\mathbf{S^1_u}(0,v) + h(v)\mathbf{S^1_v}(0,v) \tag{10}$$

Here, $k(v)$ and $h(v)$ are v related scalar functions which can be defined as eqution (11):

$$\begin{aligned}k(v) &= k_0(1-v)+k_1 v\\ h(v) &= h_0(1-v)+h_1 v\end{aligned} \tag{11}$$

and k_0, k_1, h_0, h_1 are arbitrary real numbers.

If $\mathbf{S^2_u}(0,v)$, $\mathbf{S^1_u}(0,v)$ and $\mathbf{S^1_v}(0,v)$ in Equation (10) are replaced with the corresponding partial derivative respectively, a quadratic Bernstein polynomials Equation (12) can be obtained:

$$\sum_{j=0}^{2} B_j^2(v)\mathbf{F}_j = k(v)\sum_{j=0}^{2} B_j^2(v)\mathbf{E}_j + h(v)\sum_{j=0}^{2} B_j^2(v)\mathbf{D}_j \tag{12}$$

Here,

$$\begin{aligned}\mathbf{F}_0 &= \mathbf{P}_{01}-\mathbf{P}_0\\ \mathbf{F}_1 &= (1-v)(\mathbf{q}_{02}-\mathbf{P}_{02})+v(\mathbf{q}_{20}-\mathbf{P}_{02})\\ \mathbf{F}_2 &= \mathbf{P}_{21}-\mathbf{P}_{20}\\ \mathbf{E}_0 &= \mathbf{P'}_{01}-\mathbf{P'}_0\\ \mathbf{E}_1 &= (1-v)(\mathbf{q'}_{02}-\mathbf{P'}_{02})+v(\mathbf{q'}_{20}-\mathbf{P'}_{02})\\ \mathbf{E}_2 &= \mathbf{P'}_{21}-\mathbf{P'}_{20}\\ \mathbf{D}_0 &= \mathbf{P'}_{02}-\mathbf{P'}_0\\ \mathbf{D}_1 &= \mathbf{P'}_{20}-\mathbf{P'}_{02}\\ \mathbf{D}_2 &= \mathbf{P'}_2-\mathbf{P'}_{20}\end{aligned} \tag{13}$$

$\mathbf{F}_j$, $\mathbf{E}_j$ and $\mathbf{D}_j$ are called vectors from adjacent control point pairs. $\mathbf{F}_j$, $\mathbf{E}_j$ can define the cross boundary tangent vectors, while $\mathbf{D}_j$ can define the common boundary tangent vectors.

If the coefficients of $\mathbf{F}_j$ and $\mathbf{E}_j$ in Equation (12) are changed into cubic Bernstein polynomials, we can get the following Equation (14):

$$\sum_{j=0}^{3} B_j^3(v)\mathbf{F'}_j = k(v)\sum_{j=0}^{3} B_j^3(v)\mathbf{E'}_j + h(v)\sum_{j=0}^{2} B_j^2(v)\mathbf{D}_j \quad (14)$$

Here,

$$\begin{aligned}
\mathbf{F'}_0 &= \mathbf{F}_0 \\
\mathbf{F'}_1 &= \frac{\mathbf{F}_0 + 2(\mathbf{q}_{02} - \mathbf{P}_{02})}{3} \\
\mathbf{F'}_2 &= \frac{\mathbf{F}_2 + 2(\mathbf{q}_{20} - \mathbf{P}_{02})}{3} \\
\mathbf{F'}_3 &= \mathbf{F}_2 \\
\mathbf{E'}_0 &= \mathbf{E}_0 \\
\mathbf{E'}_1 &= \frac{\mathbf{E}_0 + 2(\mathbf{q'}_{02} - \mathbf{P'}_{02})}{3} \\
\mathbf{E'}_2 &= \frac{\mathbf{E}_2 + 2(\mathbf{q'}_{20} - \mathbf{P'}_{02})}{3} \\
\mathbf{E'}_3 &= \mathbf{E}_2
\end{aligned} \quad (15)$$

In order to solve Equation (14), we make use of the differential vector expression as follows:

$$\sum_{j=0}^{n} B_j^n(v)\mathbf{P}_j = \{\mathbf{P}_0, \ldots, \mathbf{P}_n\} = \{\mathbf{P}_j\}_n \quad (16)$$

According to the property of Bernstein polynomials shown in Equation (17)

$$\begin{aligned}
v\{\mathbf{P}_j\}_n &= \{\frac{j\mathbf{P}_{j-1}}{n+1}\}_{n+1} \\
(1-v)\{\mathbf{P}_j\}_n &= \{\frac{(n+1-j)\mathbf{P}_j}{n+1}\}_{n+1}
\end{aligned} \quad (17)$$

and Equation (18)

$$\begin{aligned}
m\{\mathbf{P}_j\}_n &= \{m\mathbf{P}_j\}_n \\
\{\mathbf{a}_j\}_n + \{\mathbf{b}_j\}_n &= \{\mathbf{a}_j + \mathbf{b}_j\}_n
\end{aligned} \quad (18)$$

Equation (14) can be changed into Equation (19):

$$\begin{aligned}
\{\mathbf{F'}_j\}_3 &= (1-v)\{\mathbf{F'}_j\}_3 + v\{\mathbf{F'}_j\}_3 \\
k(v)\{\mathbf{E'}_j\}_3 &= k_0(1-v)\{\mathbf{E'}_j\}_3 + k_1 v\{\mathbf{E'}_j\}_3 \\
h(v)\{\mathbf{D}_j\}_2 &= h(v)((1-v)\{\mathbf{D}_j\}_2 + v\{\mathbf{D}_j\}_2)
\end{aligned} \quad (19)$$

Reorganizing the above Equations by using undetermined coefficient method, we can obtain the following Equation (20),

$$\begin{aligned}
\mathbf{F'}_0 &= k_0\mathbf{E'}_0 + h_0\mathbf{D}_0 \\
\mathbf{F'}_1 &= \frac{3k_0\mathbf{E'}_1 + (k_1 - k_0)\mathbf{E'}_0 + 2h_0\mathbf{D}_1 + h_1\mathbf{D}_0}{3} \\
\mathbf{F'}_2 &= \frac{3k_1\mathbf{E'}_2 + (k_0 - k_1)\mathbf{E'}_3 + 2h_1\mathbf{D}_1 + h_0\mathbf{D}_2}{3} \\
\mathbf{F'}_3 &= k_1\mathbf{E'}_3 + h_1\mathbf{D}_2
\end{aligned} \quad (20)$$

in that there is an additional limiting condition as follows:

$$\begin{aligned}
3\mathbf{F'}_1 + 3\mathbf{F'}_2 = {} & 3k_0\mathbf{E'}_2 + 3k_1\mathbf{E'}_1 + 2h_1\mathbf{D}_1 \\
& + 2h_0\mathbf{D}_1 + h_0\mathbf{D}_2 + h_1\mathbf{D}_0
\end{aligned} \quad (21)$$

In Figure (3), $\mathbf{F'}_0$, $\mathbf{F'}_3$, $\mathbf{E'}_0$, $\mathbf{E'}_3$, $\mathbf{D}_j(j = 0, 1, 2)$ are known and $\mathbf{F'}_1$, $\mathbf{F'}_2$, $\mathbf{E'}_1$, $\mathbf{E'}_2$ are unknown. Because Equation (20) is tenable for any $v \in [0, 1]$, the values of k_0, k_1, h_0 and h_1 can be determined at the end points by evaluating Equation (22) at $v = 0$ and $v = 1$:

$$\begin{aligned}
\mathbf{F'}_0 &= k_0\mathbf{E'}_0 + h_0\mathbf{D}_0 \\
\mathbf{F'}_3 &= k_1\mathbf{E'}_3 + h_1\mathbf{D}_2
\end{aligned} \quad (22)$$

By comparing Equation (20) with (21), the following ratio Equation (23) can be obtained:

$$\mathbf{E'}_3 - \mathbf{E'}_0 = 3(\mathbf{E'}_2 - \mathbf{E'}_1) \quad (23)$$

A characteristic solution of Equation (23) is:

$$\begin{aligned}
\mathbf{E'}_1 &= \frac{2\mathbf{E'}_0 + \mathbf{E'}_3}{3} \\
\mathbf{E'}_2 &= \frac{\mathbf{E'}_0 + 2\mathbf{E'}_3}{3}
\end{aligned} \quad (24)$$

Therefore,

$$\begin{aligned}
\mathbf{F'}_1 &= \frac{k_0(\mathbf{E'}_0 + \mathbf{E'}_3) + k_1\mathbf{E'}_0 + 2h_0\mathbf{D}_1 + h_1\mathbf{D}_0}{3} \\
\mathbf{F'}_2 &= \frac{k_1(\mathbf{E'}_0 + \mathbf{E'}_3) + k_0\mathbf{E'}_3 + h_0\mathbf{D}_2 + 2h_1\mathbf{D}_1}{3}
\end{aligned} \quad (25)$$

Especially, if $2\mathbf{D}_1 = \mathbf{D}_0 + \mathbf{D}_2$, the following Equation (26) is tenable.

$$\mathbf{F'}_3 - \mathbf{F'}_0 = 3(\mathbf{F'}_2 - \mathbf{F'}_1) \quad (26)$$

In other words, there is the symmetric relationship between $\{\mathbf{F'}_j\}_4$ and $\{\mathbf{E'}_j\}_4$ when $2\mathbf{D}_1 = \mathbf{D}_0 + \mathbf{D}_2$.

According to vector addition principle corresponding to Figure 3, the following Equation (27) is tenable.

$$\begin{aligned}
\mathbf{P}_{21} - \mathbf{P}_{20} &= (\mathbf{P}_{21} - \mathbf{P}_2) + (\mathbf{P}_2 - \mathbf{P}_{20}) \\
\mathbf{q}_{20} - \mathbf{P}_{02} &= (\mathbf{q}_{20} - \mathbf{P}_{20}) + (\mathbf{P}_{20} - \mathbf{P}_{02}) \\
\mathbf{P'}_{21} - \mathbf{P'}_{20} &= (\mathbf{P'}_{21} - \mathbf{P'}_2) + (\mathbf{P'}_2 - \mathbf{P'}_{20}) \\
\mathbf{q'}_{20} - \mathbf{P'}_{02} &= (\mathbf{q'}_{20} - \mathbf{P'}_{20}) + (\mathbf{P'}_{20} - \mathbf{P'}_{02})
\end{aligned} \quad (27)$$

By the substitution of $\mathbf{P}_0 = \mathbf{P'}_0$, $\mathbf{P}_{02} = \mathbf{P'}_{02}$, $\mathbf{P}_{20} = \mathbf{P'}_{20}$, $\mathbf{P}_2 = \mathbf{P'}_2$ and Equation (27) for Equation (20), the following Equation (28) can be obtained:

$$\begin{aligned}
\mathbf{F''}_0 &= k_0\mathbf{E''}_0 + h_0\mathbf{D}_0 \\
\mathbf{F''}_3 &= k_1\mathbf{E''}_3 + h_1'\mathbf{D}_2 \\
\mathbf{E''}_1 &= \frac{\mathbf{E''}_0 + \mathbf{E''}_3 + \mathbf{D}_2}{2} \\
\mathbf{E''}_2 &= \frac{\mathbf{E''}_0 + \mathbf{E''}_3 + \mathbf{D}_2 - 2\mathbf{D}_1}{2} \\
\mathbf{F''}_1 &= \frac{2k_0\mathbf{E''}_1 + k_1\mathbf{E''}_0 + h_1\mathbf{D}_0 + 2h_0\mathbf{D}_1 - \mathbf{F''}_0}{2} \\
\mathbf{F''}_2 &= \frac{2k_1\mathbf{E''}_2 + k_0\mathbf{E''}_3 + 2h_1'\mathbf{D}_1 + h_0'\mathbf{D}_2 - \mathbf{F''}_3}{2}
\end{aligned} \quad (28)$$

Here,

$$
\begin{aligned}
\mathbf{F''}_0 &= \mathbf{P}_{01} - \mathbf{P}_0 \\
\mathbf{F''}_1 &= \mathbf{q}_{02} - \mathbf{P}_{02} \\
\mathbf{F''}_2 &= \mathbf{q}_{20} - \mathbf{P}_{20} \\
\mathbf{F''}_3 &= \mathbf{P}_{21} - \mathbf{P}_2 \\
\mathbf{E''}_0 &= \mathbf{P'}_{01} - \mathbf{P}_0 \\
\mathbf{E''}_1 &= \mathbf{q'}_{02} - \mathbf{P}_{02} \\
\mathbf{E''}_2 &= \mathbf{q'}_{20} - \mathbf{P}_{20} \\
\mathbf{E''}_3 &= \mathbf{P'}_{21} - \mathbf{P}_2
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
h'_0 &= k_0 + h_0 - 1 \\
h'_1 &= k_1 + h_1 - 1
\end{aligned} \tag{30}
$$

The above process demonstrated how to determine and calculate the coordinates of internal control points when two triangular Gregory patches are adjacent with G^1 continuity.

B. Between Quadrangular and Triangular Gregory Patches

For quadrangular and triangular Gregory patches, the control nets and expression are different. A schematic drawing of a quadrangular Gregory patch $\mathbf{S^1}$ adjacent to a triangular Gregory patch $\mathbf{S^2}$ is shown in Figure 4.

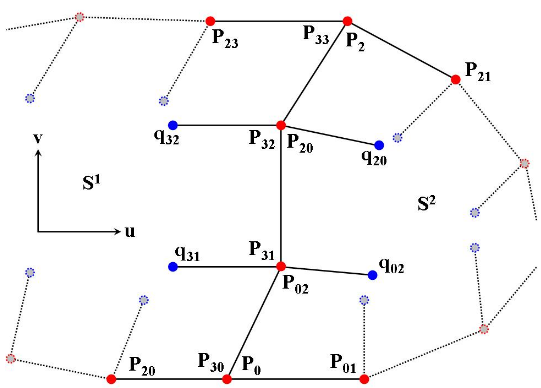

Fig. 4. Two adjacent triangular and quadrangular Gregory patches

Similarly, when $\mathbf{S_u}$ and $\mathbf{S_v}$ are corresponding to the partial derivative of u, v direction, the necessary and sufficient condition that two adjacent patches $\mathbf{S^1}$ and $\mathbf{S^2}$ satisfy the G^1 continuity can be expressed by Equation (10), too. By using Bernstein polynomials expression, the same form as Equation (14) can be obtained. It needs to be paid attention that $\mathbf{F'}_j$ and $\mathbf{D}_j$ should have the same geometry meaning as defined in Equations (13) and (15). While, $\mathbf{E'}_j$ is defined as follows:

$$
\begin{aligned}
\mathbf{E'}_0 &= \mathbf{P}_{20} - \mathbf{P}_{30} \\
\mathbf{E'}_1 &= \mathbf{q}_{31} - \mathbf{P}_{31} \\
\mathbf{E'}_2 &= \mathbf{q}_{32} - \mathbf{P}_{32} \\
\mathbf{E'}_3 &= \mathbf{P}_{23} - \mathbf{P}_{33}
\end{aligned} \tag{31}
$$

Equations (16)-(26) can also be suitable to two adjacent triangular and quadrangular Gregory patches. It is because that they have the same expression form. When $\mathbf{F''}_j$ is defined by Equation (29), the Equation (32) is tenable.

$$
\begin{aligned}
\mathbf{F''}_0 &= k_0\mathbf{E'}_0 + h_0\mathbf{D}_0 \\
\mathbf{F''}_3 &= k_1\mathbf{E'}_3 + h'_1\mathbf{D}_2 \\
\mathbf{E'}_1 &= \frac{2\mathbf{E'}_0 + \mathbf{E'}_3}{3} \\
\mathbf{E'}_2 &= \frac{\mathbf{E'}_0 + 2\mathbf{E'}_3}{3} \\
\mathbf{F''}_1 &= \frac{k_0\mathbf{E'}_3 + k_1\mathbf{E'}_0 + (h'_1 - h'_0)\mathbf{D}_0 + 2h_0\mathbf{D}_1}{2} \\
\mathbf{F''}_2 &= \frac{k_1\mathbf{E'}_0 + k_0\mathbf{E'}_3 + 2h'_1\mathbf{D}_1 + (h'_0 - h'_1)\mathbf{D}_2}{2}
\end{aligned} \tag{32}
$$

Here,

$$
\begin{aligned}
h'_0 &= h_0 - 1 \\
h'_1 &= h_1 - 1
\end{aligned} \tag{33}
$$

Equations (31)-(33) demonstrat how to determine and calculate the coordinates of internal control points when one quadrangular Gregory patch and one triangular Gregory patch are connected together with G^1 continuity.

IV. Conclusion and Future Work

In this paper, we have proposed a new mathematical expression of bi-cubic triangular Gregory patch. Comparing with traditional triangular Gregory patch, it's bi-cubic in the true sense. Then, we derive the necessary and sufficient condition that two adjacent triangular Gregory patches satisfy the G^1 continuity, and how to determine the coordinates of internal control points is demonstrated. At last, we have briefly discussed the G^1 continuity condition between quadrangular and triangular Gregory patches, and how to determine the coordinates of internal control points is also demonstrated.

In the future work, we will apply new triangular Gregory patch to solve the N-side problem, lathy triangular region interpolation problem, 3D shape design and reconstruction problems.

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